KALIKADEVI ART'S, COMMERCE & SCIENCE COLLEGE, SHIRUR(KA)
Department of Mathematics

REAL ANALYSIS

DEPARTMENT OF MATHEMATICS
MR.GHADGE R.B

METRIC SPACE

- **DEFINITION:***METRIC*

Let X be a non empty set. Define $d: X \times X \to \mathbb{R}$.

Then d is said to be a metric on X if it satisfies the following conditions

- Non negativity:

$d(x,y) \geq 0$ for all $x, y \in X$

- Definiteness:

$d(x, y)=0$ iff $x=y$

- Symmetry:

$d(x,y) = d(y,x)$

- Triangle inequality:

$d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$

- **DEFINITION:** *METRIC SPACE*

A non empty set X along with the metric d on X is called Metric space. It is denoted by (X, d) or simply X.

- **Example:**

Let $X = \mathbb{R}$.

Define $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x, y)=|x - y|$ for all $x, y \in \mathbb{R}$

To check: d is a metric on $\mathbb{R}$.

- Non negativity:

$d(x,y) = |x - y| \geq 0$

- Definiteness:

$d(x, y) = 0 \Leftrightarrow |x - y| = 0$

$\Leftrightarrow x - y = 0$

$\Leftrightarrow x = y$

- Symmetry:

$d(x, y) = |x – y|$

$= |-(y – x)|$

$= |y- x|$

$= d(y , x)$

- Triangle inequality:

Consider $d(x,y) = |x- y|$

$=| x –z + z – y |$

$= |(x-z) – (y-z)|$

$= |(x-z) + (z-y)|$

$\leq |x-z| + |z-y|$

$=d(x,z) + d(z,y)$ for all $x, y, z \in \mathbb{R}$

Since d satisfies all the axioms of the metric.

Hence d is the metric on ℝ called *Usual Metric* or *Standard Metric*.

Definition: *NEIGHBOURHOOD OF A POINT*

Let (X,d) be a Metric space.

Let p ∈X , then a neighbourhood of a point p is a set N_r(p) defined as N_r(p) = { x ∈ X : d(x,y) < r}. The number r > 0 is called radius of N_r(p).

- **DEFINITION**: INTERIOR POINT

Let (X , d) be a metric space. Let E⊂ X. Let p ∈ E.

Then p is said to be an interior point of E if there exists a neighbourhoodN_r(p) such that N_r(p) ⊂ E.

EXAMPLE:

Let X = ℝ

E = ℤ = {…, -2,-1,0,1,2, …}⊂ ℝ.

W.K.T Neighbourhood of x =N_x(x) = (x-ϵ , x+ϵ)

For ϵ = 2,The neighbourhood of -1 = (-3, 1) ⊄ ℤ.

That is, we cannot find any neighbourhood of any integer which is contained in ℤ.

Therefore ℤ has no interior points.

- **EXAMPLE OF INTERIOR POINT**:

Let E = (2 , 6) ⊂ ℝ.

All reals between 2 and 6 are interior points.

- **DEFINITION**: OPEN SET

Let (X , d) be a metric space. Let E ⊂X. then E is said to be an open set if every point of E is an interior point.

- **EXAMPLE OF OPEN SET:**

Let X = ℝ, with usual metric.

- Let E = (-1 , 5)

Points of E = all reals between -1 and 5

Interior point of E = all reals between -1 and 5

Therefore E is open.

In General, Every open interval is open.

- Let E = (0, 5) ∪ {7} ⊂ ℝ

Points of E = all reals between 0 and 5, 7

Interior points of E = all reals between 0 and 5.

Therefore E is not open.

- Let E = [1 , 5]⊂ ℝ

Points of E = 1, 5 and all reals between 1 and 5

Interior points of E = all reals between 1 and 5.

Therefore E is not open.

Theorem:

Every neighbourhood is an open set.

Proof:

Let (X , d) be a Metric space.

Let p ∈X. Let N_r(p) be an arbitrary neighbourhood of p, r > 0.

To prove: N_r(p) is an open set.

i.e., to prove : Every point of N_r(p) is an interior point.

Let q ∈ N_r(p)

To prove: q is an interior point of N_r(p) .

Let d(p, q) = h (h > 0, h < r) …………..(1)

Let N_{r-h}(q) be a neighbourhood of q.

To prove: $N_{r-h}(q) \subseteq N_r(p)$

Let x ∈ N_{r-h}(q)

Now to prove: x ∈ N_r(p)

now x ∈ $N_{r-h}(q) \Rightarrow d(x, q) < r - h$ ………..(2)

Consider d(x , p) ≤ d(x , q) + d(q , p)

(by triangle inequality)

< r – h + h = r (by (1) & (2))

d(x , p) < r ⇒ x ∈ N_r(p)

Therefore we have N_{r-h}(q) ⊆ N_r(p)

i.e., there exists a neighbourhood of q which is contained in N_r(p).

⇒ q is an interior point of N_r(p).

Since q is arbitrary, every point of N_r(p) is an interior point.

N_r(p) is an open set and

since this is an arbitrary neighbourhood.

We can say that every neighbourhood is an open set.

DEFINITION: LIMIT POINT OF A SET

Let (X , d) be a metric space. Let E$\subset$ X. Let p $\in$ X.

Then p is said to be a limit point of E if every neighbourhood of p contains atleast one point of E other than p.

DEFINITION: DERIVED SET

The set of all limit points of E is called derived set of E. It is denoted by E'.

EXAMPLE:

- Let E = (-1 , 10]

Limit points of E = -1,10 and all reals between -1 and 10.

E' = { -1,10, all reals between -1 and 10}

= [-1, 10]

- Let E = (1,5) ∪ {5}

Limit point of E = 1,5, all reals between 1 and 5.

DEFINITION: CLOSED SET

Let (X , d) be a metric space. Let E$\subset$ X.

Then E is said to be closed set if every limit point of E is a point of E.

EXAMPLE:

- Let E = [0 , 1] $\subset \mathbb{R}$

Limit points of E = 0, 1, all reals between 0 and 1.

Points of E = 0, 1, all reals between 0 and 1.

Therefore E is closed.

In General, every closed interval is a closed set.

DEFINITION: COMPLEMENT OF A SET

Let (X , d) be a metric space. Let E$\subset$ X.

The complement of E is denoted by E^c and is defined as $E^c = \{ x \in X : x \notin E \}$

EXAMPLE:

- Let ($\mathbb{R}$, d) be a Metric space.

$\mathbb{R}^c = \emptyset$, $\emptyset^c = \mathbb{R}$

THE RELATION BETWEEN OPEN AND CLOSED SETS

Theorem:

A set E is open iff its complement is closed.

Proof:

Necessary part:

Let E be open.

To prove: E^c is closed.

Let p be a limit point of E^c.

It is enough to show that p $\in E^c$.

Since p is a limit point of E^c, every neighbourhood of p contain at least one point of E^cother than p.

⇒ No neighbourhood of p is contained in E.

⇒ p is not an interior point of E.

But E is open.

⇒ $p \notin E$

⇒ $p \in E^c$

Since p is arbitrary,we can say every limit point is a point of E^c.

Therefore E^c is closed.

Sufficient part:

Suppose E^c is closed.

To prove: E is open.

Let p ∈E (arbitrary).

To prove: p is an interior point of E.

Since p ∈E which implies p ∉ E^c.

But E^c is closed.

⇒ p is not a limit point of E^c.

⇒ there exists a neighbourhood N of p which contains no point of E^c.

Which implies that ∃ a neighbourhood N of p such that $N \cap E^c = \emptyset$

i.e., ∃ a neighbourhood N of p such that

$N \subset (E^c)^c = E$

⇒ ∃ a neighbourhood N of p such that $N \subset E$

⇒ p is an interior point of E.

Since p is arbitrary, every point of E is an interior point of E.

Therefore E is open.

Thank You