# T.Y.B.SC. PHYSICS

# Fifth Semester Paper XV

## Classical & Quantum Mechanics


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# CHAPTER - 1

# CLASSICAL MECHANICS

Classical or Newtonian mechanics appeared on the $17^{th}$ century and was very successful in describing the physics of the macroscopic world. However, in the beginning of the 20th century, experiments revealed a series of atomic and subatomic phenomena that could not be explained by the old theory. This triggered the development of Quantum mechanics.In classical mechanics a system can be described by a number of dynamic variables. The goal is to be able to calculate the exact values of these variables at any given time and determine how these values evolve as a function of time $t$. Once the position of a particle within an one-dimensional system is known, a series of other variables such as the velocity, acceleration, momentum, potential and kinetic energy can be calculated. Newton's Law is the basis of classical mechanics:

$$F = m.a = \frac{dp}{dt}$$

**1.2 : Mechanics of particle:** We apply Newtonian mechanics to deduce conservation laws for a particle in motion. These laws tell us under what conditions the mechanical quantities like linear momentum, angular momentum, energy etc.are constant in time.

## a) Conservation of Linear Momentum:

If a force F is acting on a particle of mass m, then according to Newton's second law of motion, we have

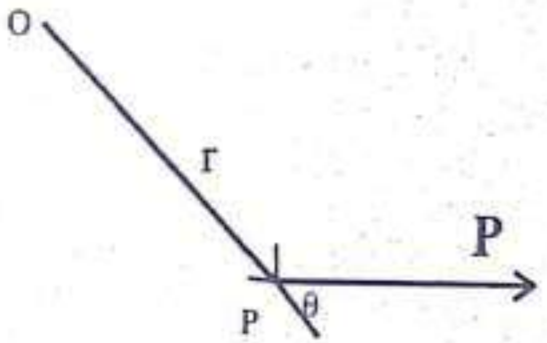
$$F = m.a = \frac{dp}{dt} = \frac{d}{dt}(m.v)$$

where p = mv- is the linear momentum of the particle.
If the external force, acting on the particle is zero, then

$$\frac{dp}{dt} = \frac{d}{dt}(m.v) = 0$$

Thus in absence of external force, the linear momentum of a particle is conserved. This is the conservation theorem for a free particle.

**b) Conservation of Angular Momentum:**




The angular momentum of a particle P of a mass in about a point O is defined as

$$J = r \times p \qquad \ldots\ldots \qquad (1)$$

where r- is the position vector of the particle P
p=mv- is its linear momentum.
If the force on the particle is F, then the moment of force or torque about 0 is defined as

$$J = r \times p \qquad \ldots\ldots \qquad (1)$$

where r- is the position vector of the particle P

p=mv- is its linear momentum.

If the force on the particle is F, then the moment of force or torque about 0 is defined as

$$\tau = r \times F$$

If we differentiate (1) with respect to t, then

$$\frac{dJ}{dt} = \frac{d}{dt}(r \times p) = \frac{dr}{dt} \times p + r \times \frac{dp}{dt}$$

$$\frac{dJ}{dt} = r \times F \left[ \frac{dr}{dt} \times p = v \times mv = 0 \, and \, F = \frac{dp}{dt} \right]$$

$$\therefore \frac{dJ}{dt} = \tau$$

i.e the rate of change of angular momentum of a particle is equal to the torque acting on it.Now, if the torque acting on the particle is zero.i.e. $\tau$ =0

$$\frac{dJ}{dt} = 0 \text{ or } J=0$$

Thus in absence of external torque, theangular momentum of a particle is conserved. This is the conservation theorem of angular momentum of particle.

c) **Conservation of Energy:**

i) **Work :** Work done by an external force F upon a particle in displacing from point 1 to point 2

$$W_{12} = \int_1^2 F.dr \qquad \text{----- (1)}$$

ii) **Kinetic Energy and Work-Energy Theorem :**

According to Newton's second law, F = m.dv/dt and hence

$$F.dr = m\frac{dv}{dt}.dr = m\frac{dv}{dt}.vdt\left[\because dr = m\frac{dr}{dt}dt = v.dt\right]$$

$$= m\frac{d}{dt}\left[\frac{1}{2}v.v\right]dt$$

$$= d\left[\frac{1}{2}mv^2\right]$$

Therefore, equation (1) become,

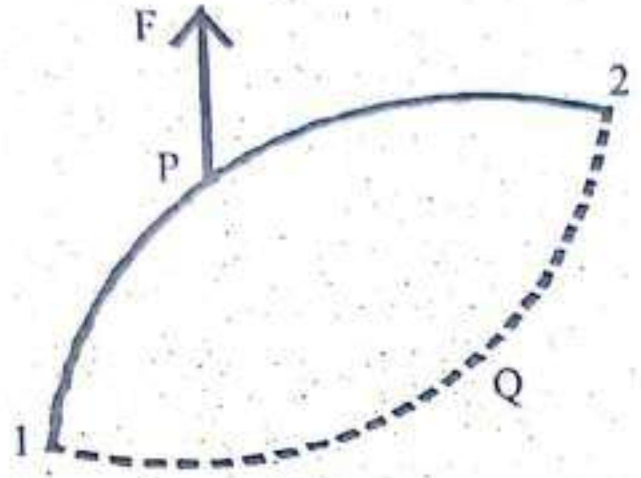
$$W_{12} = \int_1^2 F.dr = \int_1^2 d\left[\frac{1}{2}mv^2\right] = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

The scalar quantity $\frac{1}{2}mv^2$ is defined as the kinetic energy and denoted by T. Thus the work done by the force acting on the particle appears equal to the change in the kinetic energy i.e.,

$$W_{12} = \int_1^2 F.dr = T_2 - T_1$$

This is known as work-energy theorem.

### iii) Conservative Force and Potential Energy:




[ Fig. Work done by a force on a particle. ]

If the work done ($W_{12}$) by the force in moving a particle from point 1 to point 2 is the same for any possible path between the points, then the force (and the system) is said to be conservative. The region in which the particle is experiencing a conservative force is called as conservative force field.

Thus for conservative force

$$P\int_1^2 F.dr = Q\int_1^2 F.dr$$

$$P\int_1^2 F.dr + Q\int_2^1 F.dr = 0$$

$$\int F.dr = 0 \qquad \text{------ (1)}$$

Thus, if the force is consevative, the work done on the particle around a closed path in the force field is zero.

According to stoke's theorem in vector analysis, we can tranform the equation (1) as

$$\oint F.dr = \iint Curl\ F.ds \qquad \text{------ (2)}$$

Since the work done is zero around any closed path in the conservative force field and does not depend on the length of the path, we may carry out the integration over the perimeter of the area ds.

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This gives,

$$\oint F.dr = \iint Curl\ F.ds = 0 \qquad \text{------ (3)}$$

But, $ds \neq 0$ and hence in general,

$$Curl\ F = 0 \ or\ \nabla \times F = 0 \quad ...(24)$$

Therefore the force can be expressed as,

$$F = -\nabla V = -\left( \vec{i}\frac{\partial V}{\partial x} + \vec{j}\frac{\partial V}{\partial y} + \vec{k}\frac{\partial V}{\partial z} \right) \qquad \text{------ (4)}$$

because,

$$\nabla \times \nabla V = \vec{i}\left( \frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y} \right) + \vec{j}\left( \frac{\partial^2 V}{\partial z \partial x} - \frac{\partial^2 V}{\partial x \partial z} \right) + \vec{k}\left( \frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x} \right) = 0$$

This scalar function V is called the potential or potential energy and depends on position. In case, if we add any constant quantity to V, equation (4) does not change and hence the zero or reference level of the potential function V is arbitrary and can be chosen at convenience.

If we take scalar product of dr with (4) and integrate from position 1 to position 2, we obtain,

$$\int_1^2 F.dr = -\int_1^2 \nabla V.dr = -\int_1^2 dV = V_1 - V_2$$

Now, if we assume the position 1 as $\infty$ and the potential energy to be zero there, then the potential energy at a point r (position 2) is given by

$$V(r) = -\int_{\infty}^{r} F.dr$$

The work done by the conservative force is

$$W_{12} = \int_1^2 F.dr = V_1 - V_2$$

which is the change in potential energy when the particle moves from 1 to 2

**iv) Conservation theorem:**

According to work energy theorem, the amoun t of work done by a force in moving a particle from 1 to 2 is given by the change in kinetic energy. i.e.

$$W_{12} = \int_1^2 F.dr = T_2 - T_1 \qquad \text{------ (1)}$$

Also, we know that the formula for work done by the conservative force is,

$$W_{12} = \int_1^2 F.dr = V_1 - V_2 \qquad \text{------ (2)}$$

From (1) and (2) we get,

$$V_1 - V_2 = T_2 - T_1$$

$$T_1 + V_1 = T_2 + V_2 = const.$$

Thus the sum of kinetic and potential energy of particle remains constant in a conservative force field. This is known as the law of conservation of energy.

$$m.\frac{dv}{dt} = F$$

If we multiply by v=dr/dt to both sides and integrate with respect to t.
we obtain,

$$\int m.\frac{dv}{dt}.vdt = \int F.\frac{dr}{dt}.dt + Const.(say\, E)$$

$$\int \frac{d}{dt}\left[\frac{1}{2}mv.v\right]dt = \int F.dr + E$$

$$\int \frac{d}{dt}\left[\frac{1}{2}mv^2\right]dt - \int F.dr = E$$

$$\frac{1}{2}mv^2 - \int_{\infty}^{r} F.dr = E$$

$$T + V = E$$

It represents the conservation energy theorem

## 1.3 : Mechanics of system of particle:

Newton's third law of motion, equal and opposite forces, does not hold for all forces. It is called the weak law of action and reaction.

Center of mass :

$$R = \frac{\sum m_i r_i}{\sum m_i} = \frac{\sum m_i r_i}{M}$$

Center of mass moves as if the total external force were acting on the entire mass of the system concentrated at the center of mass. Internal forces that obey Newton's third law, have no effect on the motion of the center of mass.

$$F^{(e)} = M\frac{d^2R}{dt^2} = \sum_i F_i^{(e)}$$

Motion of center of mass is unaffected.

**Total linear momentum :**

$$P = \sum_i m_i \frac{dr_i}{dt} = M\frac{dR}{dt}$$

**Conservation Theorem for the Linear Momentum of a System of Particles:**

If the total external force is zero, the total linear momentum is conserved. The strong law of action and reaction is the condition that the internal forces between two particles, in addition to being equal and opposite, also lie along the line joining the particles. Then the time derivative of angular momentum is the total external torque:

$$\frac{dL}{dt} = N^{(e)}$$

Torque is also called the moment of the external force about the given point. Conservation Theorem for Total Angular Momentum: L is constant in time if the applied torque is zero. inear Momentum Conservation requires weak law of action and reaction. Angular Momentum Conservation requires strong law of action and reaction.

**Total Angular Momentum:**

$$L = \sum_i r_i \times p_i = R \times Mv + \sum_i r_i' \times p_i'$$

Total angular momentum about a point O is the angular momentum of motion concentrated at the center of mass, plus the angular momentum of motion about the center of mass. If the center of mass is at rest w.r.t. the origin then the angular momentum is independent of the point of reference.

**Total Work:**

$W_{12} = T_2 - T_1$, where T is the total kinetic energy of the system: $T = \frac{1}{2}\sum_i m_i v_i^2$

**Total kinetic energy:**

$$T = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}mv^2 + \frac{1}{2}\sum_i m_i v_i'^2$$

Kinetic energy, like angular momentum, has two parts: the K.E. obtained if all the mass were concentrated at the center of mass, plus the K.E. of motion about the center of mass.

**Total potential energy:**

$$V = \sum_i v_i + \frac{1}{2}\sum_{\substack{i,j \\ i \neq j}} v_{ij}$$

If the external and internal forces are both derivable from potentials it is possible to define a total potential energy such that the total energy T + V is conserved.

The term on the right is called the internal potential energy. For rigid bodies the internal potential energy will be constant. For a rigid body the internal forces do no work and the internal potential energy remains constant

## 1.4 Constraints :

Often the motion of a particle or system of particles is restricted by one or more conditions. The limitations on the motion of a system are called constraints and the motion is said to be constrained motion

**Classification of Constraints:** *Constraints are classified as*,

1) **Scleronomic :** If constraint relations do not explicitly depend on time.

   *e.g.* rigid body.

2) **Rheonomic:** If constraint relations depend on time.

   *e.g.* A bead sliding on a moving wire.

3) **Holonomic:** If constraint relations are made independent of velocity.

   *e.g.* A cylinder rolling without sliding down an inclined plane.

4) **Non - holonomic:** If constraint relations are not holonomic, that is these relations are irreducible functions of velocities.

   *e.g.* sphere rolling without sliding down an inclined plane.

5) **Bilaleral:** If the constraint relations are expressed in the form of equations.

   *e.g.* rigid body.

6) **Unilateral:** If the constraint relations are expressed in the form of inequalities.

*e.g.* Motion of molecules in a gas container or Motion of particle on the surface of a sphere under the action of gravity one time rolling on the surface of sphere & other time leaving the surface $\left(r^2 \geq a^2\right)$.

7) **Conservative:** If Forces of constraint do not do any work & total mechanical energy of the system is conserved while performing the constraint motion.

*e.g.* Simple pendulum with rigid support.

8) **Dissipative:** If the forces of constraint do work & the total mechanical energy is not conserved.

*e.g.* Pendulum with variable length.

### 1.4.1 Holonomic constraints:

The nomenclature 'holonomic' constraints comes from the word 'holos' which means integer in Greek and 'whole or integrable' in latin languages. Constraints limit the motion of a system and the number of independent coordinates, needed to describe the motion, is reduced. For example, if a particle is allowed to move on the circumference of a circle, then only one coordinate $q_1 = \theta$ is sufficient to describe the motion, because the radius $(a)$ of the circle remains the same. If r is the position vector of the particle at any angular coordinate $\theta$ relative to the centre of the circle, then

$$|r| = a \quad or \quad r - a = 0 \qquad \ldots\ldots \qquad (1)$$

Eq. (1) expresses the constraint for a particle in circular motion. Similarly in the case of a particle, moving on the surface of a sphere, the correct coordinates are spherical coordinates r, $\theta$ and $\phi$, where $\theta$ and $\phi$ only vary. Therefore $q_1 = \theta$ and $q_2 = \phi$ are the two independent coordinates for the problem, because the constraint is that the radius of the sphere (a) is constant (i.e., r = a). Since in the circular motion of the particle, one independent coordinate 0 is needed, the number of degrees of freedom of the system is 1. For the partide, constrained to move on the surface of the sphere, two independent coordinates specify its motion and hence the degrees of freedom is 2.

Suppose the constraints are present in the system of N particles. If the constraints are expressed in the form of equations of the form

$$f(r_1, r_2, r_3, \ldots t) = 0$$

then they are called holonomic constraints. Let there be m number of such equations to describe the constraints in the N particle system. Now, we may use these equations to eliminate m of the 3N coordinates and we need only n independent coordinates to describe the motion, given by

$$n = 3N - m$$

The system is said to have n or $3N - m$ degrees of freedom. The elimination of the dependent coordinates can be expressed by introducing $n = 3N - m$ independent variables $q_1, q_2, \ldots q_n$ These are referred as generalized coordinates.

**Superfluous Coordinates :** The idea of degrees of freedom makes it clear that when we are using, say rectangular cartesian coordinates, we have several redundant or superfluous coordinates, if there are holonomic constraints. This redundance and non-independence of the coordinates makes the problem complicated and this difficulty is resolved by using the generalized coordinates. For example, let us consider a body be thrown vertically upward with an initial velocity $v_0$. The body will move in a straight line. In Cartesian coordinates, the motion will be represented as

$$x = 0, y = v_0 t - \frac{1}{2} g t^2, z = 0$$

where X and Z axes are horizontal and Y-axis is in vertical direction. At different values of the time t, only y coordinate varies and x and z coordinates remain the same. Therefore x and z coordinates are superfluous coordinates. In conclusion, we need only one coordinate to describe the vertical motion,

**Some more example of holonomic constraints :**

a) **Rigid body :** In case of the motion of a rigid body the distance between any two particles of the body remains fixed and do not change with time. If $r_i$ and $r_j$ are the postion vector of the ith and jth particles, then the distance between them can be expressed by the condition

$$\left| r_i - r_j \right| = r_{ij} (const.)$$

If $(x_i, y_i, z_i)$ and $(x_j, y_j, z_j)$ are the cartesian coordinates of the two particles,then the constraints will be expressed as

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = C_{ij}^{\ 2} \qquad \ldots\ldots \qquad (2)$$

This constraint is called holonomic and scleronomic.

**b) Simple pendulum with rigid support :**

In case of a simple pendulum with rigid support, the constraint is that during the motion, the distance($l$) of the bob from the point of suspension, then the condition of constraints can be expressed as,

$$|r| = l(const.)$$

This is also called as holonomic and scleronomic

## 1.4.2. Nonholonomic constraints:

The constraints which are not expressible in the form of eq. (2) are called nonholonomic. For example, the motion of a particle, placed on the surface of a sphere of radius a, will be described by

$$|r| \geq a \ \ or \ \ r - a \geq 0$$

in a gravitational field, where r is the position vector of the particle relative to the centre of the sphere. The particle will first slide down the surface and then fall off. The gas molecules in a container are constrained to move inside it and the related constraint is another example of nonholonomic constraints. If the gas container is in spherical shape with radius a and r is the position vector of a molecule, then the condition of constraint for the motion of molecules can be expressed as

$$|r| \leq a \ \ or \ \ r - a \leq 0$$

It is to be mentioned that in holonomic constraints, each coordinate can vary independently of the other. In a notholonomic system, all the coordinates cannot vary independently and hence the number of degrees of freedom of the system is less than the minimum number of coordinates needed to specify the configuration of the system.

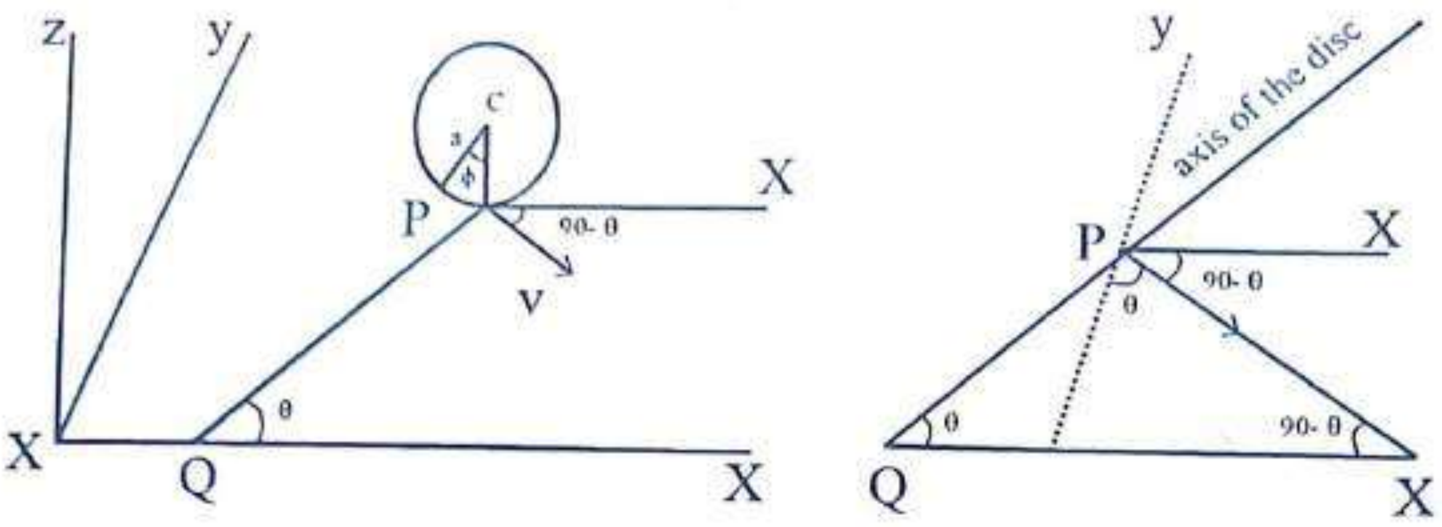
Constraints are further described as *(i) rheonomous and (ii) scierononious.*

In the former, the equations of constraint contain the time as an explicit variable, while in the later they are not explicitly dependent on time. Constraints may also be classified as

(i) conservative and (ii) dissipative. In case of conservative constraints, total mechanical energy of the system is conserved during the constrained motion and the constraint forces do not do any work. In dissipative constraints, the constraint forces do work and the total mechanical energy is not conserved. Time-dependent or rheonomic constraints are generally dissipative.

**Example of non holonomic constraints:**

**Rolling disc :** A system is said to be non-holonomic if it corresponds to non-integrable differential equations of constraints. Such constraints cannot be expressed in the form of eq.(3). Obviously holonomic system has integrable differential equations of constraints, expressible in the form (3). In order to explain this, let us consider a disc rolling on a rough horizontal X-Y plane with the condition of constraint is that the plane of the disc is always vertical. We choose the coordinates x, y for the centre of the disc, for the angle of rotation about the axis of the disc and $\theta$ for the angle between the axis of the disc and X-axis




**[ Fig. : Vertical disc rolling on a horizontal XY-plane ]**

If a is the radius of the disc, the constraint that the axis of the disc is perpendicular to the vertical direction, gives the velocity v of the disc with magnitude

$$v = a\phi = a\frac{d\phi}{dt}$$

As the direction of the velocity is perpendicular to the axis of the disc, the components of the velocity along X-axis and Y-axis are

$$v_x = \frac{dx}{dt} = v\sin\theta \quad , \quad v_y = \frac{dy}{dt} = -v\cos\theta$$

$$\frac{dx}{dt} = a\frac{d\phi}{dt}\sin\theta \quad and \quad \frac{dy}{dt} = -a\frac{d\phi}{dt}\cos\theta$$

$$dx - a\sin\theta d\phi = 0 \quad and \quad dy + a\cos\theta d\phi = 0 \qquad \ldots\ldots \qquad (4)$$

None of the equations, given by (4), can be integrated without solving the entire problem. Thus the constraint cannot be put in the form $f(r_1, r_2, r_3, \ldots t) = 0$ and hence the constraint is nonholonomic.

## 1.5 Principal of virtual work:

In order to investigate the properties of a system, we can imagine arbitrary instantaneous change in the position vectors of the particles of the system e.g., virtual displacements. An infinite virtual displacement of $i^{th}$ particle of a system of N particles is denoted by $\delta r_i$. This is the displacement of position coordinates only and does not involve variation of time i.e.,

$$\delta r_i = \delta r_i(q_1, q_2, \ldots q_n)$$

Suppose the system is in equilibrium, then the total force on any particle is zero i.e.,

$$F_i = 0 \quad i = 1,2,\ldots N$$

The virtual work of the force $F_i$ in the virtual displacement $\delta r_i$, will also be zero i.e.,

$$\delta W_i = F_i \cdot \delta r_i = 0$$

Similarly, the sum of virtual work for all the particles must vanish i.e.,

$$\delta W = \sum_{i=1}^{N} F_i \cdot \delta r_i = 0$$

This result represents the principle of virtual work which states that the work done is zero in the case of an arbitrary virtual displacement of a system from a position of equilibrium

The total force $F_i$ on the ith particle can be expressed as

$$F_i = F_i^a + f_i$$

where $F_i^a$ is the applied force and $f_i$ the force of constraint.

Hence eq. (11) assumes the form

$$\sum_{i=1}^{N} F_i \cdot \delta r_i + \sum_{i=1}^{N} f_i \cdot \delta r_i = 0$$

We restrict ourselves to the systems where the virtual work of the forces of constraints is zero, e.g., in case of a rigid body. Then

$$\sum_{i=1}^{N} F_i \cdot \delta r_i = 0$$

For equilibrium of a system, the virtual work of applied forces is zero.

### 1.6. D'Alembert's Principle :

According to Newton's second law of motion, the force acting on the $i^{th}$ particle is given by

$$F_i = \frac{dp_i}{dt} = \dot{P}_i$$

This can be written as prove

$$F_i - \dot{P}_i = 0 \qquad i = 1,2,\ldots N$$

These equations mean that any particle in the system is in equilibrium under a force, which is equal the actual force $F_i$ plus a reversed effective force $\dot{P}_i$. Therefore, for virtual displacements $\delta r_i$,

$$\sum_{i=1}^{N}\left(F_i - \dot{P}_i\right)\delta r_i = 0$$

But $F_i = F_i^a + f_i$ then

$$\sum_{i=1}^{N}\left(F_i^a - \dot{P}_i\right)\delta r_i + \sum_{i=1}^{N} f_i.\delta r_i = 0$$

Again, we restrict ourselves to the systems for which the virtual work of the constraints is zero,

$$\sum_{i=1}^{N} f_i.\delta r_i = 0$$ Then,

$$\sum_{i=1}^{N}\left(F_i^a - \dot{P}_i\right)\delta r_i = 0$$

This is known as D'Alembert's principle. Since the forces of constraints do not appear in the equation and hence now we can drop the superscript. Therefore, the D'Alembert's principle may be written as principle may be written as

$$\sum_{i=1}^{N}\left(F_i-\dot{P}_i\right)\delta r_i=0$$

## 1.7 Lagrange's equations from D'Alembert's Principle :

Consider a system of N particles. The transformation equations for the position vectors of the particles are

$$r_i=r_i(q_1,q_2,\ldots\ldots,q_k,\ldots q_n,t) \qquad \text{----- (1)}$$

where t is the time and $q_k$ (k =1, 2,...,n) are the generalized coordinates.

Differentiating eq. (1) with respect to t, we obtain the velocity of the $i^{th}$ particle, i.e.,

$$\frac{dr_i}{dt}=\frac{\partial r_i}{\partial q_1}\frac{dq_1}{dt}+\frac{\partial r_i}{\partial q_2}\frac{dq_2}{dt}+\ldots..+\frac{\partial r_i}{\partial q_k}\frac{dq_k}{dt}+\ldots..+\frac{\partial r_i}{\partial q_n}\frac{dq_n}{dt}+\frac{\partial r_i}{\partial t}$$

$$v_i=\dot{r}_i=\sum_{k=1}^{n}\frac{\partial r_i}{\partial q_k}\dot{q}_k+\frac{\partial r_i}{\partial t} \qquad \text{----- (2)}$$

where $\dot{q}_k$ are the generalized velocities.

The virtual displacement is given by

$$\delta r_i = \frac{\partial r_i}{\partial q_1}\delta q_1 + \frac{\partial r_i}{\partial q_2}\delta q_2 + \ldots\ldots + \frac{\partial r_i}{\partial q_k}\delta q_k + \ldots\ldots + \frac{\partial r_i}{\partial q_n}\delta q_n$$

or

$$\delta r_i = \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k}\delta q_k \qquad \text{------ (3)}$$

Since by definition the virtual displacements do not depend on time.

According to D'Alembert's principle,

$$\sum_{i=1}^{N} \left(F_i - \dot{P}_i\right)\delta r_i = 0 \qquad \text{----- (4)}$$

Here

$$\sum_{i=1}^{N} F_i \delta r_i = \sum_{i=1}^{N} F_i \cdot \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k}\delta q_k = \sum_{k=1}^{n}\sum_{i=1}^{N}\left[F_i \cdot \frac{\partial r_i}{\partial q_k}\right]\delta q_k = \sum_{k=1}^{n} G_k \delta q_k \qquad \text{----- (5)}$$

$$G_k = \sum_{i=1}^{N} F_i \cdot \frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{N} F_{x_i}\frac{\partial x_i}{\partial q_k} + F_{y_i}\frac{\partial y_i}{\partial q_k} + F_{z_i}\frac{\partial z_i}{\partial q_k} \qquad \text{------ (6)}$$

are called the components of generalized force associated with the generalized coordinates $q_k$ This may be mentioned that as the dimensions of the generalized coordinates need not be those of length, similarly the generalized force components $G_k$ may have dimensions different than those of force. However, the dimensions of $G_k \delta q_k$ are those of work.

Further,

$$\sum_{i=1}^{N} P_i.\delta r_i = \sum_{i=1}^{N} m_i r_i.\sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k}\delta q_k = \sum_{k=1}^{n}\left[\sum_{i=1}^{N} m_i r_i.\frac{\partial r_i}{\partial q_k}\right]\delta q_k \quad \text{------ (7)}$$

$$\sum_{i=1}^{N} m_i \ddot{r}_i.\frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i \dot{r}_i.\frac{\partial r_i}{\partial q_k}\right) - m_i \dot{r}_i.\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right)\right] \quad \text{------ (8)}$$

It is easy to prove that,

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) = -\frac{\partial}{\partial q_k}\left(\frac{dr_i}{dt}\right) = \frac{\partial v_i}{\partial q_k} \quad \text{------ (9)}$$

$$\frac{\partial r_i}{\partial q_k} = \frac{\partial v_i}{\partial q_k} \quad \text{------ (10)}$$

$$\sum_{i=1}^{N} m_i \ddot{r}_i.\frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i v_i.\frac{\partial v_i}{\partial q_k}\right) - m_i v_i.\left(\frac{\partial v_i}{\partial q_k}\right)\right] \quad \text{------ (11)}$$

$$= \sum_{k=1}^{n}\left[\frac{d}{dt}\left\{\frac{\partial}{\partial q_k}\left(\sum_{i=1}^{N}\frac{1}{2}m_i(v_i.v_i)\right)\right\} - \frac{\partial}{\partial q_k}\left(\sum_{i=1}^{N}\frac{1}{2}m_i(v_i.v_i)\right)\right].\delta q_k$$

$$= \sum_{k=1}^{n}\left[\frac{d}{dt}\left\{\frac{\partial T}{\partial q_k}\right\} - \frac{\partial T}{\partial q_k}\right].\delta q_k \qquad \text{----- (12)}$$

Here

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) = \sum_{j=1}^{n}\frac{\partial^2 r_i}{\partial q_j \partial q_k}\dot{q}_j + \frac{\partial^2 r_i}{\partial t \partial q_k} \qquad \text{----- (13)}$$

which has been obtained by treating $\frac{\partial r_i}{\partial q_k}$ as a single quantity being the function of the generalized coordinates $q_j$ and time t.

but, $$v_i = \frac{dr_i}{dt} = \sum_{j=1}^{n}\frac{\partial r_i}{\partial q_j \partial q_k}\dot{q}_j + \frac{\partial r_i}{\partial t}$$

and its partial derivative with respect to $q_k$ is

$$\frac{\partial v_i}{\partial q_k} = \frac{\partial}{\partial q_k}\left(\frac{dr_i}{dt}\right) = \sum_{j=1}^{n} \frac{\partial^2 r_i}{\partial q_j \partial q_k}\dot{q}_j + \frac{\partial^2 r_i}{\partial t \partial q_k} \qquad \text{------ (14)}$$

From equation (13) and (14) $\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) = \frac{\partial v_i}{\partial q_k}$

$$\frac{\partial v_i}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}\left(\frac{dr_i}{dt}\right) = \frac{\partial}{\partial \dot{q}_k}\left[\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}\right] = \frac{\partial r_i}{\partial q_j}\delta_{jk} = \frac{\partial r_i}{\partial q_k} \quad \text{... (15)}$$

as the constraints are holononiic and $\frac{\partial \dot{q}_j}{\partial \dot{q}_k} = \delta_{jk}$ is kronecker delta which is 1 for j=k

and zero for $j \neq k$ Substituting for $\sum_{i=1} F_i \delta r_i$ from (6) and $\sum_{i=1} \dot{P}_i . \delta r_i$ from (12) in eq. (4), the D,Alembert,s principle becomes,

$$\sum_{k=1}^{n}\left\{\left[\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_k}\right\} - \frac{\partial T}{\partial q_k}\right] - G_k\right\}.\delta q_k = 0$$

$$\left[\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_k}\right\}-\frac{\partial T}{\partial q_k}\right]=0 \quad \text{or} \quad \left[\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_k}\right\}-\frac{\partial T}{\partial q_k}\right]=G_k \quad \text{------ (16)}$$

This forms as general form of Lagrange's equations.

For a conservative system, the force is derivable from a scalar potential V.

$$F_i=\nabla V_i=-\left(\vec{i}\,\frac{\partial V}{\partial x_i}+\vec{j}\,\frac{\partial V}{\partial y_i}+\vec{k}\,\frac{\partial V}{\partial z_i}\right)$$

Hence from eq. (6), the generalised force component are

$$G_k=\sum_{i=1}^{N} F_i.\frac{\partial r_i}{\partial q_k}=\sum_{i=1}^{N}\left(\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_k}+\frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_k}+\frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_k}\right)$$

Clearly the right hand side of equation is the partial derivatives of -V with resect to $q_k$

$$G_k=-\frac{\partial V}{\partial q_k} \qquad \text{------ (17)}$$

$$\left[\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_k}\right\}-\frac{\partial T}{\partial q_k}\right]=-\frac{\partial V}{\partial q_k}$$

$$\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_k}\right\}-\frac{\partial T}{\partial q_k}+\frac{\partial V}{\partial q_k}=0$$

$$\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_k}\right\}-\frac{\partial (T-V)}{\partial q_k}=0 \qquad \text{------ (18)}$$

Since the scalar potential V is the function of generalized coordinates $q_k$ only not depending on generalized velocites, we can write eq.(18) as

$$\frac{d}{dt}\left\{\frac{\partial (T-V)}{\partial \dot{q}_k}\right\}-\frac{\partial (T-V)}{\partial q_k}=0 \qquad \text{------ (19)}$$

We define a new function given by

L=T-V

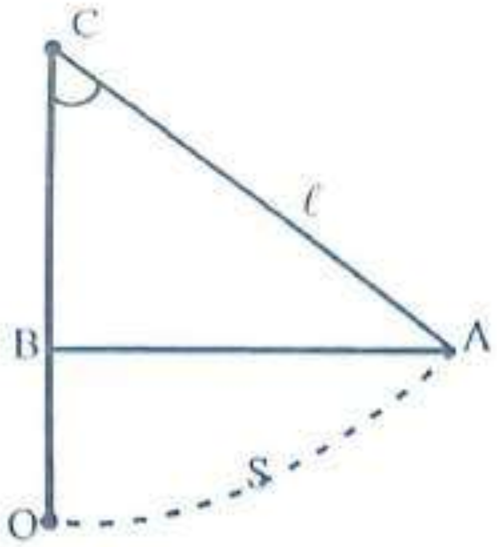
which is called the Lagrangians of the system. This equation(19) takes the form

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_k}\right\}-\frac{\partial L}{\partial q_k}=0$$

This equation is known as Lagrange's equation for the conservative system

## Simple application of Lagrange's formulation:

### a) Simple pendulum:




**[ Fig. 2.5 Simple pendulum ]**

Let $\theta$ be the angular displacement of the simple pendulum from the equilibrium position. If $l$ be the effective length of the pendulum and m be the mass of the bob, then the displacement along arc OA=s is given by

$$s = l\theta \qquad \because \theta = \frac{Arc}{Radius} = \frac{s}{l}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2 \qquad \because v = \frac{ds}{dt} = \frac{d}{dt}(l\theta) = l\frac{d\theta}{dt} = l\dot{\theta}$$

If the potential energy of the system, when the bob is at 0, is zero, then the potential energy, when the bob is at A, is given by

$V = mg(OB) = mg(OC - BC) = mg(l - l\cos\theta) = mgl(1 - \cos\theta)$

$$L = T - V \quad or \quad L = \frac{1}{2} ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

Now, $\frac{\partial L}{\partial \theta} = -mgl\sin\theta \quad and \quad \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$

Substituting these values in the Lagrange's equation (here there is only one generalized coordinate $q_1 = \theta$)

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\theta}}\right\} - \frac{\partial L}{\partial \theta} = 0$$

We get,

$$\frac{d}{dt}\left[ml^2\dot{\theta}\right] + mgl\sin\theta = 0 \quad or \quad ml^2\ddot{\theta} + mgl\sin\theta = 0$$

or $$\ddot{\theta}+\frac{g}{l}\sin\theta=0$$

This represents the equation of motion of a simples pendulum.

For small amplitude oscillations, $\sin\cong\theta$ , and therefore the equation of motion of a simple pendulum is,

$$\ddot{\theta}+\frac{g}{l}\theta=0$$

This represents a simple harmonic motion of period, given by

$$T=2\pi\sqrt{\frac{l}{g}}$$

**b) Particle in space:**

**i) Motion of one particle using cartesian coordinates:**

The generalized forces needed in $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial \dot{q}_j}=Q_j$ are obviously $F_x,\ F_y\ and\ F_z$. Then

$$\frac{\partial T}{\partial x}=\frac{\partial T}{\partial y}=\frac{\partial T}{\partial z}=0$$

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}\ , \quad \frac{\partial T}{\partial \dot{y}} = m\dot{y}\ , \quad \frac{\partial T}{\partial \dot{z}} = m\dot{z}$$

and the equation of motion are

$$\frac{d}{dt}(m\dot{x}) = F_x, \quad \frac{d}{dt}(m\dot{y}) = F_y, \quad \frac{d}{dt}(m\dot{z}) = F_z$$

We are thus led back to the original Newtons equations of motion.

**ii) Motion of one particle using polar coordinates:**

We must express T in terms of $\dot{r}$ *and* $\dot{\theta}$. The transformation equations are,

$$x = r\cos\theta$$

$$y = r\sin\theta$$

The velocities are given by,

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

The kinetic energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ then reduces formally to

$$T = \frac{1}{2} m\left[\dot{r}^2 + (r\dot{\theta})^2\right]$$

An alternative derivation of above equation is obtained by recognizing that the polar components of the velocity are $\dot{r}$ along $\mathbf{r}$, and $r\dot{\theta}$ along the direction perpendicular to r, denoted by the unit vector $\hat{\boldsymbol{\theta}}$. Hence, the square of the velocity expressed in polar coordinates is simply $\left[\dot{r}^2 + (r\dot{\theta})^2\right]$.

$$d\mathbf{r} = \hat{\mathbf{r}}dr + r\hat{\boldsymbol{\theta}}d\theta + \hat{\mathbf{k}}dz$$

for the differential position, $d\mathbf{r}$ ,in cylindrical coordinates, restricted to plane Z=0 where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors in r and $\theta$ direction, respectively, the components of the generalized force can be obtained from the defination $Q_\theta = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$

Therefore,

$$Q_r = \mathbf{F}.\frac{\partial \mathbf{r}}{\partial r} = \mathbf{F}.\hat{\mathbf{r}} = F_r$$

$$Q_\theta = \mathbf{F}.\frac{\partial \mathbf{r}}{\partial \theta} = \mathbf{F}.r\hat{\boldsymbol{\theta}} = rF_\theta$$

Since the derivative of r with respective to $\theta$ is, by thedefination of a derivative, a vector in the direction of $\hat{\theta}$. There are two generalized coordinates and therefore two Lagrange equations. The derivatives occuring in the r equation are,

$$\frac{\partial T}{\partial r} = mr\dot{\theta}^2, \quad \frac{\partial T}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) = m\ddot{r}$$

and the equation itself is,

$$m\ddot{r} - mr\dot{\theta}^2 = F_r$$

The second term being the centripetal acceleration term. For $\theta$ equation, we have the derivatives,

$$\frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

So that the equation becomes

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = rF_\theta$$

**c) Linear Harmonic Oscillator :**

Let us consider the motion in the direction of x axis. The kinetic energy of the harmonic oscillator is given by

$$T = \frac{1}{2}m\dot{x}^2 \quad \dot{x} = \frac{dx}{dt} = Velocity\ of\ the\ oscillator$$

and the potential is

$$V = \int F.dx$$

WhereF=-kx is the restoring force acting on a body and k is force constant, so

$$V = \int -kx.dx = \int kx.dx$$

or $$V = \frac{kx^2}{2} \quad or \quad V = \frac{1}{2}kx^2$$, where the constant of intergration has been set equal to zero by choosing v = 0 at x = 0

Thus lagrangian

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

which gives, $\frac{\partial L}{\partial \dot{x}} = \frac{1}{2} m 2\dot{x} = m\dot{x}$ ------ (1)

and $\frac{\partial L}{\partial x} = -\frac{1}{2} k 2x = -kx$

For this case

Lagrangian equation is given by

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

On substituting equation (1) and (2) in this equation we get,

$$\frac{d}{dx}(m\dot{x}) - kx = 0$$

or $m\ddot{x} + kx = 0$ *where* $\ddot{x}$ *means*

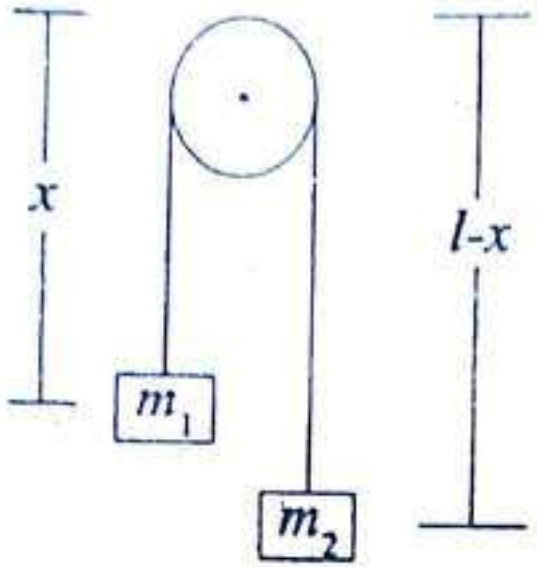
$$\frac{d^2x}{dt^2} = \frac{d}{dt}(\dot{x}) = \ddot{x}$$

$$m\ddot{x} + kx = 0$$

Which is required equation of motion of one dimensional harmonic oscillator.

**d) Atwood's Machine :**

Figure shows an example of a conservative system with holonomic, scleromous constraint (the pully is assumed frictionless and massless). Clearly there is only one independent coordinates x, the position of the other weight being determined by the constraint that the length of the rope between them is $l$. The potential energy is




**[Fig. : Atwood's machine ]**

$$V = m_1 g x - m_2 g(l - x)$$

while the kinetic energy is

$$T = \frac{1}{2}(M_1 + M_2)\dot{x}^2$$

Combining the two, Lagrangian has the form

$$L = T - V = \frac{1}{2}(M_1 + M_2)\dot{x}^2 + m_1 g x + m_2 g(l - x)$$

There is only one equation of motion, involving the derivatives

$$\frac{\partial L}{\partial x} = (M_1 - M_2)g \qquad \frac{\partial L}{\partial \dot{x}} = (M_1 + M_2)\dot{x}$$

So that we have,

$$(M_1 + M_2)\ddot{x} = (M_1 - M_2)g$$

$$\ddot{x} = \frac{(M_1 - M_2)g}{(M_1 + M_2)}$$

which is simılliar result obtained by more elementry means. This trivial problem emphasizes that the forces of constraint-here the tension in the rope appear nowhere in the Lagrangian formulation. By the same token, neither can the tension in the rope be found directly by the lagrangian method.

## :: Multiple Choice Question ::

1) The branch of physics which deals with details of motion of the point like and rigid or deformable extended object is called ............
   a) Quantum mechanics.
   **b) Classical mechanics.**
   c) Statistical mechanics.
   d) None of these.

2) The momentum of particle is constant ...........
   a) In the presence of external forces on a particle.
   **b) In the absence of external forces on a particle.**
   c) In the absence of internal forces on a particle.
   d) None of these.

3) When is the position vector and is the linear momentum of the particle at the given instant the angular momentum of the partilee is ...........
   **a)** $\overline{L} = \overline{r} \times \overline{p}$
   b) $\overline{L} = \overline{r} . \overline{p}$
   c) $\overline{L} = \overline{r}\ \overline{p}$
   d) None of these

4) The rate of change of angular momentum is ...........
   **a) Torque.**
   b) Moment of inertia.
   c) moment of momentum.
   d) None of these.

5) If no torque is acting a particle then its angular momentum is ...........
   **a) Constant.**
   b) variable.
   c) zero.
   d) None of these.

6) According to the law of conservation of linear momentum if the total external force acting on the system is equal to zero then the total linear momentum is .....

a) **Conserved.** b) not conserved.
c) constant. d) None of these.

7) If the total external torque acting on a system of particle is zero then the total angular momentum is constant is called as ..........

a) Law of conservation of angular momentum.
b) Law of conservation of linear momentum.
c) Law of conservation of total linear momentum.
d) **Law of conservation of total angualar momentum.**

8) Constaints applied in a system ..........

a) **Reduce the number of degree of freedom.**
b) Increase the number of degree of freedom.
c) Equal to number of degree of freedom.
d) None of these.

9) The constaints involved when a particle is restricted to move along a curve of surface are ..........

a) **Only one co-cordinates.** b) only two coordinates.
c) Both (a) and (b). d) None of these.

10) The constaints involved when a particle is restricted to move along a curve of surface are ..........

a) **Holonomic.** b) Non holonomic.
c) Both (a) and (b). d) None of these.

11) $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}=0$ konwn as ..........

a) **Lagrsngains eq. of motion for a conservative system.**
b) Newton's equation of motion.
c) Hamiltonion equation of motion.
d) None of these.

12) If a system is in static equllibrium then virtual work done by all the applied force vanishes $\sum_i F_i^a dr_i = o$ is ..........

a) **Principle of virtual work.** b) Principle of superposition.
c) Doppler principle. d) None of these.

13) The relation $\sum \bar{F}_i^{a} - \dot{\bar{P}}_i d\bar{r}_i = 0$ is called ......

a) Dopplers Principle.

**b) D'Alemberts Principle.**

c) Lagranges Principle.

d) None of these.

14) A particle of mass m moves in a conservative force field which of the following is not Lagrangian.

a) $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$

b) $\frac{d}{dt}\frac{\partial L}{\partial \dot{Z}} - \frac{\partial L}{\partial Z} = 0$

**c)** $\boldsymbol{\frac{d}{dt}\frac{\partial \theta}{\partial L} - \frac{\partial \theta}{\partial L} = 0}$

d) None of these

15) Lagrangains equation are applicable when the system is ..........

a) Conservative.

b) Non canservative.

**c) Both (a) and (b)**

d) None of these.

16) The force of constraints obays ..........

a) Newtons graviational law.

b) Einsteins relativity.

**c) Newtons third law of motion.**

d) friction.

17) Consider the sliding of a beed on a circularwire of radius a in the xy plane. The equation of constraint is then the constraint is known as ..........

a) Hot integrable constraint.

b) Non holonomic constraint.

**c) Holonomic constraint.**

d) Virtual constraint.

18) For a conservative system the Lagrangian equation of motion in terms of generalised co-cordinates q and momentum P is ..........

**a)** $\boldsymbol{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial p_j} = 0}$

b) $\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_j} = 0$

c) $\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_j} = 0$

d) $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$

18) Atwoods machine is an example of ........ system.

a) Linear

b) angular

**c) Conservative**

d) None of these.

19) $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial \dot{q}_k} = 0$ represents Lagrangains equation in .........

a) General system.
b) Linear system.
c) **Conservative.**
d) None of these.

20) $\delta w = \sum_{i=1}^{N} F_i^a \delta r_i = 0$ represents............

a) D'Alemberts Principle.
b) **Virtual work done.**
c) Lagrangain equation.
d) None of these.

# THANK YOU