KALIKADEVI ART’S, COMMERCE & SCIENCE COLLEGE, SHIRUR(KA)

DEPARTMENT OF MATHEMATICS

A Boundary Value Problem of Fractional Order Solutions (B.V.P.S.)

Mr. Ramesh B. Ghadge
Assistant Professor in Mathematics

Department of Mathematics,
Kalikadevi Arts, Commerce & Science College,
Shirur (Ka) Beed, Maharashtra – 413249
E-Mail: ***rameshgadge555@gmail.com***

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Introduction

Fractional Calculus is ordinary differentiation and integration.

- Fractional boundary value problem

$$D^{\propto} - (D^{\propto} + u(t)) + u(t) = ((t_1 u(t)), t \in (0, +\infty)$$

$$u(0) = u(+\infty) = 0$$

where

$$\frac{1}{2} < \propto 1 \; and \; f; (0 + \infty) \times \mathbb{R} \to \mathbb{R}$$

Definition Let μ be a function defined on (0, +∞). For

N – 1 ≤ α < n (n ∈ N*), the left and right Riemann – Liouville fractional derivatives for a function μ denoted by $D^{\alpha}{+}\mu$ and $D^{\alpha} - \mu$ respectively, are defined by

$$D^{\propto}{+}\mu(t) = \frac{d^n}{dt^n} I^{n-\alpha}{+}u(t)$$

$$= \frac{1}{T\,(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\propto-1} \mu(s)ds, t \in (0, +\infty),$$

And

$$D^{\propto}\text{-}u(t) = (-1)^n \frac{d^n}{dt^n} I^{n-\propto}\, u(t)$$

$$= \frac{(-1)^n}{T(n-\propto)} \frac{d^n}{dt^n} \int_t^{+\infty} (s-t)^{n-\propto-1}\, u(s)\, ds,\ t \in (0, +\infty),$$

Provided that the right – hand side is pointwise defined.

In particular for α=n, $D^{\propto} + u\,(t) = D^{\propto}u(t) = D^n u(t) and\ D^{\propto} - u(t) = (-1)^n\, D^n u(t),$

$(0, +\infty)$

Theorem: If $D^{\propto}+u(t)=D^{\propto}-u\in L^1(0,+\infty) and\ n-1\leq\propto<n, then$

$$I^{\propto}+D^{\propto}+u(t)=u(t)+\sum_{j=1}^{n}Cj\,(t-a)^{\propto-j}$$

$$I^{\propto}-D^{\propto}-u(t)=u(t)+\sum_{j=1}^{n}Cj\,(b-t)^{\propto-j}$$

With $C_j^1+\frac{(-1)^{\propto-1}D_{b-}^{\propto-1}}{T(\propto-j+1}\in R, j=1,2\ldots,n.$

Now we introduce a new space which is suitable for the study of our fractional BVP.

Proof: Let.

$$E_0^{\propto}(0,+\infty)=\{u\in L^2(0,+\infty).\,D^{\propto}+u\in L^2(0,+\infty),u(0)=u(\infty)=0\},$$

With the natural norm

$$||u||\propto=\left(\int_0^{+\infty}|u(t)|^2dt+\int_0^{+\infty}|D^{a}+u(t)|^2dt\right)^{\frac{1}{2}},\forall u\in E_0^{\propto}(0,+\infty).\ (1.2)$$

Let the space $C_p([0,+\infty))be\ defined\ by$

$$C_p([0,+\infty))=\{u\in C([0,+\infty)),R):\lim p(t)u(t)exists\}$$

And endowed with the norm

$$||u||\infty,p=\sup_{t\in[0,+\infty)}p(t)|u(t)|,\qquad \lim_{n\to+\infty}p(t)t^{\propto-\frac{1}{2}}=0.$$

Where the function p :[0, +∞) → (0, +∞) is continuous and satisfies

We put

$$M=\frac{1}{\sqrt{2\propto-1}.\,\Gamma(\propto)}.\sup_{t>0}p(t)t^{\propto-\frac{1}{2}}.$$

Formula:

The boundary value problem:

$$\int_0^{+\infty} [D^{\propto} + u(t)D^{\propto} + u(t) + u(t)u(t) - f(t, u(t))u(t)]dt = 0, for\ all\ u \in E_0^{\propto}(0, +\propto).$$

Detinition Let A : X → X^* be an operator on the real Banach space X.

(a) A is said to be demicontinuous if

$$u_n \rightarrow u \; as \; n \rightarrow \; +\infty \; implies \; Au_n \rightarrow Au \; as \; n \rightarrow +\infty.$$

(b) A is said to be hemicontionus if the real function.

t→ $\langle A(u + tu), w\rangle$ is continuous on [0.1] for al u, v, $w \in X$.

(c) A is said to be coercive if

$$||u||^{lim} \rightarrow +\infty \; \frac{\langle Au.u\rangle}{||u||} = \; +\infty.$$

Lemma The operator

$$\text{T} : E_0^{\propto}(0,+\infty) \to T\left(E_0^{\propto}(0,+\infty)\right) \subset L^2(0,+\infty) = L_2^2(0,+\infty)$$

$$u \to T(u) = (u, D^{\propto} + u)$$

Is an isometric isomorphic mapping.

Proof. It is clear that T is a linear operatior and we now show that T conserves norms, i.e.

$$\forall u \in E_0^{\propto}(0,+\infty): \left|\left|Tu\right|\right|L_2^2 = \left|\left|u\right|\right| \propto.$$

Indeed, we have

$$||(\text{u},D^{\propto} + u)\ ||L_2^2 = ||u|| \propto$$

Theorem: $E_0^{\propto}(0,+\infty)$ is a separable space.

Proof. Since, L^2 $(0,+\infty),\mathbb{R})$ is a separable Banach space, the Cartesian space

$$L_2^2(0,+\infty)\mathbb{R}) = L^2\ (0,+\infty)\mathbb{R}) \times L^2\ (0,+\infty)\mathbb{R})$$

Is also a separable Banach space with respect to the norm.

$||u||L_2^2 \sum_{i=1}^{2} \left|\left|u_i\right|\right| L^2 where\, u = (u_1, u_2) \in L_2^2(0,+\infty)\mathbb{R}).$

Then, the space T$(E_0^{\propto}(0,+\infty)) \subset L_2^2$ is also separable

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$$\text{T}: E_0^{\propto}(0,+\infty) \to T(E_0^{\propto}(0,+\infty) \subset L_2^2(0,+\infty)$$

$$\text{u} \to T(u) = (u, D^{\propto} + u)$$

is an isometric isomorphic, so $E_0^{\alpha}(0,+\infty)$ is a separable space.

Lemma For all u∈ $E_0^{\propto}(0,+\infty)$ we have that $E_0^{\propto}(0,+\infty)$ embeds continuously in $C_p\ (0,+\infty))$.

$$M_0 > 0, \left|\left|u\right|\right|_{\infty,p} \leq M_0 \left|\left| u\right|\right|_{\propto}.$$

Proof. For all u∈ $E_0^{\propto}(0,+\infty), and\ t\ > 0,$

U(t)=$I_+^{\propto}\left(D^{\propto} + u(t)\right),$

So

P(t)u(t)=p(t) $I_+^{\propto}\left(D^{\propto} + u(t)\right)$

Which implies from the Cauchy-Schwartz inequality

$$\left|p(t)I_+^{\propto}\left(D^{\propto} + u(t)\right)\right| = \frac{p(t)}{T(\propto)}\left| \int_0^t (t-s)^{\propto-1} D^{\propto} + u(s)ds\right|$$

$$\leq \frac{p(t)}{T(\propto)}(\int_0^t (t-s)^{2(\propto-1)}ds)^{\frac{1}{2}}(\int_0^t D^{\propto} + u(s))^2 ds)^{\frac{1}{2}}$$

$$\leq \frac{p(t)}{T(\propto)}(\int_0^t (t-s)^{2(\propto-1)}ds)^{\frac{1}{2}}(\int_0^{+\infty} |u(s)|^2 ds$$

$$+ \int_0^{+\infty} |D^{\propto} + u(s)|^2 ds)^{\frac{1}{2}}$$

$$= \frac{||u|| \propto}{\sqrt{2 \propto -1.T(\propto)}} p(t) t^{\propto - \frac{1}{2}}$$

Then

$$||u||\infty, p = \sup_{t \in [0,+\infty)} |p(t)u(t)|$$

$$= \sup_{t \in [0,+\infty)} |p(t) I_+^{\propto}(D^{\propto} + u(t))|$$

$$\leq \frac{||u||\propto}{\sqrt{2\propto-1.T(\propto)}} . \sup_{t>0} p(t) t^{\propto - \frac{1}{2}}, \qquad ||u||\infty, p \leq M||u|| \propto.$$

From the definition of the norm in $E_0^{\propto}(0, +\infty)$, it is easy to see that

Theorem The embedding

$$E_0^{\propto}(0,+\infty) \to C_p([0,+\infty))$$

Is compact.

Proof. Let D $\subset$ $E_0^{\propto}(0,+\infty)$ be a bounded set. Then it is bounded in

$C_p([0,+\infty))$ by Lemma . Let R> 0 be such that for all $u \in D \|u\|_{\propto} \leq R$.

We will apply Lemma

(a) D is equicontinuous on every compact interval of $[0,+\infty)$.

Let $u \in D$ *and* $t_1, t_2 \in J \subset [0,+\infty)$. where J is a compact sub-interval and by the Cauchy Schwarz inequality, we have

$$|p(t)I^{\propto} + u(t1) - p(t_2)I^{\propto} + u(t_2) = \frac{1}{T(\propto)} | p(t_1) \int_0^{t_1} (t_1 - s)^{\propto - 1} u(s) ds$$

$$- \quad p(t_2) \int_0^{t_1} (t_1 - s)^{\propto - 1} u(s) ds \,|$$

$$\leq \frac{1}{T(\propto)} | p(t_1) \int_0^{t_1} (t_1 - s)^{\propto - 1} u(s) ds$$

$$- \quad p(t_2) \int_0^{t_1} (t_1 - s)^{\propto - 1} u(s) ds \,|$$

$$+ \frac{p(t_2)}{T(\propto)} | \int_{t_1}^{t_2} (t_2 - s)^{\propto - 1} u(s) ds|$$

$$\leq \frac{1}{T(\propto)} \int_0^{t_1} | p(t_1) (t_1 - s)^{\propto - 1} p(t_2)(t_2 - s)^{\propto - 1})||u(s)|ds$$

$$+\frac{p(t_2)}{T(\propto)}\left|\int_{t_1}^{t_2}(t_2-s)^{\propto-1}u(s)ds\right|$$

$$\leq \frac{||u||L^2}{T(\propto)}[(\int_0^{t_1}(p(t_2)(t_1-s)^{\propto-1}-p(t_2)(t_2-s)^{\propto-1})^2ds)^{1/2}$$

$$- \quad p(t_2)\ \int_{t_1}^{t_2}(t_2-s)^{2\propto-2}ds]$$

$$|p(t_1)u(t_1)-\ |p(t_2)u(t_2)|=p(t_1)I^{\propto}+D^{\propto}+u(t_1)-\ p(t_2)I_0^{\propto}+D^{\propto}+u(t_2)|$$

$$<\frac{||D^{\propto}+u||L^2}{T(\propto)}\left(\int_0^{t_1}(p(t_1)(t_1-s)^{\propto-1}-p(t_2)(t_2-s)^{\propto-1})^2ds)^{1/2}\right)$$

$$+\frac{||D^{\propto}+u||L^2}{T(\propto)}\,p(t_2)\left(\int_{t_1}^{t_2}(t_2-s)^{2\propto-2}ds\right)^{1/2}$$

$$\leq \frac{R}{T(\propto)}\left(\int_0^{t_1}(p(t_1)(t_1-s)^{\propto-1}-p(t_2)(t_2-s)^{\propto-1})^2ds)^{1/2}\right)$$

$$+\frac{R}{T(\propto)}\,p(t_2)\ (\int_{t_1}^{t_2}(t_2-s)^{(\propto-1)}ds)^{\frac{1}{2}}\ \to\ 0.$$

As $|t_1-t_2|\ \to 0.$

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